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The Mermin–Wagner Phenomenon and Cluster Properties of One- and Two-Dimensional Systems

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We give optimal conditions concerning the range of interactions for the absence of spontaneous breakdown of continuous symmetries for one- and two-dimensional quantum and classical lattice and continuum systems. For a class of models verifying infrared bounds our conditions are necessary and sufficient. Using the same techniques we obtain "*a priori*" bounds on clustering for systems with continuous symmetry, improving results of Jasnow and Fisher.

KEY WORDS: Symmetry breaking; cluster properties.

1. INTRODUCTION

There are by now several papers[(1-5)] proving absence of spontaneous breakdown of continuous symmetry for one- and two-dimensional systems at nonzero temperature, for not too long-range interactions. This is what we call the Mermin–Wagner phenomenon, as the basic ideas (and in some cases even the techniques) are already present in their original papers,[(6,7)] where the absence of spontaneous magnetization was proved for the quantum and classical Heisenberg models.

In this paper we obtain best possible results concerning the range of the interaction for the absence of spontaneous breakdown of continuous

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symmetries in one- and two-dimensional systems. In fact, our condition $\int_{|p|<\epsilon} E(p)^{-1} d^\nu p = \infty$ for a suitable function $E(p)$ depending on the range of the interaction, is also necessary for a class of systems satisfying infrared bounds and sum rules. As proved in Ref. 8 for instance, for these systems the condition $\int[1/E(p)]d^\nu p < \infty$ implies breakdown of continuous symmetry.

The basic ideas of our proofs are borrowed from Ref. 1, and our contribution consists in giving optimal conditions for the validity of the arguments in Ref. 1 and also in showing that the same ideas can be used in proving cluster properties of certain correlation functions. This last result is a sharper version of results by Jasnow and Fisher.[9]

Our results apply to classical and quantum systems both in the lattice and in the continuum cases, and the only property of the equilibrium state we use is Bogoliubov's inequality. In particular we do not assume either translation invariance of the state or of the interaction.

We also prove cluster properties of the type

$$|\langle A(0)B(x)\rangle|^2 \leqslant \text{const}\left[\int \frac{1-\cos p.x}{E(p)}\, d^\nu p\right]^{-1}$$

for A (or B) of the form $A=[J,C]$ for some local C where J is an infinitesimal generator of the symmetry group. These bounds are an improved version of results by Fisher and Jasnow[9] as they are pointwise with no assumption about the "sign" of the interaction: they depend only on the range of forces and include many body interactions.

Our bounds are so to speak "*a priori*" as the rate of decay of correlation functions are model and temperature independent. Better (temperature-dependent) bounds are, of course, possible as for instance in Ref. 10, but they will be model dependent.

In the Appendix we prove some estimates of independent interest as they extend results of Ref. 8 concerning the infrared behavior of one- and two-dimensional lattice systems with long-range interactions.

2. LATTICE SYSTEMS

2.1. Absence of Symmetry Breaking

We begin with lattice systems in order to make our ideas more transparent. Our system is described as usual[11] by a C^* algebra of observables $\mathcal{Q} = \overline{\bigcup_{\Lambda\subset\mathbb{Z}^\nu}\mathcal{Q}_\Lambda}$ where the union is taken over all bounded subsets Λ of $\mathbb{Z}^\nu$, $\nu = 1$ or 2; the bar indicates norm closure and $\mathcal{Q}_\Lambda$ is the C^*

algebra of observables in Λ. The continuous symmetry group is described by a one-parameter group of automorphisms $\{\sigma_s, s \in \mathbb{R}\}$ of $\mathcal{Q}$ such that $\sigma_s \mathcal{Q}_\Lambda = \mathcal{Q}_\Lambda$. A state ω is said to be invariant under the symmetry group if $\omega(\sigma_s A) = \omega(A)$, $\forall A \in \mathcal{Q}$ and $s \in \mathbb{R}$. This is equivalent to

$$\frac{d}{ds}\omega(\sigma_s A)\bigg|_{s=0} = 0, \qquad \forall A \in \mathcal{Q}$$

If we assume the existence of local generators $J(x) \in \mathcal{Q}_{\{x\}}$, $x \in \mathbb{Z}^\nu$ such that

$$\frac{d}{ds}(\sigma_s A)\bigg|_{s=0} = i[J_\Lambda, A], \qquad \forall A \in \mathcal{Q}_\Lambda$$

where $J_\Lambda = \sum_{x \in \Lambda} J(x)$, then the invariance of ω is equivalent to

$$\omega([J_\Lambda, A]) = 0, \qquad \forall A \in \mathcal{Q}_\Lambda \quad \text{and all } \Lambda$$

For each $x \in \mathbb{Z}^\nu$ let $\sigma_s(x)$ be the action of the symmetry group at the site x. Following an idea of L. Landau (see Ref. 1), for a given function $f: \mathbb{Z}^\nu \to \mathbb{R}$ we define

$$\sigma_s(f) = \bigotimes_{x \in \mathbb{Z}^\nu} \sigma_{sf(x)}(x) \tag{2.1}$$

The only property of an equilibrium state we are going to assume is Bogoliubov's inequality, which may then be written in the form[12]

$$\left|\frac{d}{ds}\omega(\sigma_s(f)A)\bigg|_{s=0}\right|^2 \leqslant \beta\omega\left(\frac{A^*A + AA^*}{2}\right)\omega(K) \tag{2.2}$$

where $K = (d/ds)(d/dt)[\sigma_s(f)\sigma_t(f)H]|_{s=0,\,t=0}$. From (2.1)

$$\omega(K) = \sum_{x,y \in \mathbb{Z}^\nu} f(x)f(y)j(x,y) \tag{2.3}$$

where $j(x,y) = (d/ds)(d/dt)\omega(\sigma_s(x)\sigma_t(y)H)|_{s=0,\,t=0}$. In terms of local generators $\omega(K) = \sum_{x,y \in \mathbb{Z}^\nu} f(x)f(y)\omega([J(x),[J(y),H]])$.

The assumptions on H and f we are going to make (see Section 2.3) will ensure that $\omega(K)$ is well defined by (2.3) provided $f(x) \in l^1(\mathbb{Z}^\nu)$.

Theorem 1. Let $j(x,y)$ satisfy the following properties:

(i) $j(x,y) = j(y,x)$.

(ii) $j(x,.) \in l^1(\mathbb{Z}^\nu)$ and $\sum_{y \in \mathbb{Z}^\nu} j(x,y) = 0$, $\forall x \in \mathbb{Z}^\nu$.

(iii) There exists a function $g \in l^1(\mathbb{Z}^\nu)$ such that $|j(x,y)| \leqslant g(x-y)$, $\forall x, y \in \mathbb{Z}^\nu$, and

$$\int_{|p|<\delta} \frac{d^\nu p}{E(p)} = \infty \qquad \text{for all} \quad \delta > 0 \tag{2.4}$$

where

$$E(p) = \sum_{x \in \mathbb{Z}^\nu} (1 - \cos p.x) g(x) \tag{2.5}$$

Then the state ω satisfying Bogoliubov's inequality (2.2) is invariant under the symmetry group.

Proof. Let $A \in \mathscr{Q}_\Lambda$, for some finite $\Lambda \subset \mathbb{Z}^\nu$. For simplicity let us take $0 \in \Lambda$. Then

$$\frac{d}{ds}\,\omega(\sigma_s A)\Big|_{s=0} = \frac{(d/ds)\omega(\sigma_s(f)A)}{f(0)}\Big|_{s=0} \tag{2.6}$$

for any $f: \mathbb{Z}^\nu \to \mathbb{R}$ such that $f(x) = f(0) \neq 0$, $\forall x \in \Lambda$. Bogoliubov's inequality then reads

$$\left|\frac{d}{ds}\,\omega(\sigma_s A)\Big|_{s=0}\right|^2 \leqslant \beta\omega\left(\frac{A^*A + AA^*}{2}\right)\frac{\omega(K)}{|f(0)|^2} \tag{2.7}$$

From (i), (ii), and (iii),

$$\begin{aligned} |\omega(K)| &\leqslant \frac{1}{2}\sum_{x,y \in \mathbb{Z}^\nu} [f(x) - f(y)]^2 |j(x,y)| \\ &\leqslant \frac{1}{2}\sum_{x,y \in \mathbb{Z}^\nu} [f(x) - f(y)]^2 g(x-y) = \int_{B_\nu} \frac{d^\nu p}{(2\pi)^\nu}\,|\tilde{f}(p)|^2 E(p) \end{aligned} \tag{2.8}$$

with the Fourier transform $\tilde{f}(p)$ given by

$$\tilde{f}(p) = \sum_{x \in \mathbb{Z}^\nu} e^{-ip.x} f(x) \quad \text{and} \quad B_\nu = [-\pi, \pi]^\nu$$

Without loss of generality (see Remark at the end of the proof) we can assume that there are $\gamma > 0$ and $\delta > 0$ such that

$$E(p) \geqslant \gamma |p|^2 \qquad \text{for} \quad |p| \leqslant \delta \tag{2.9}$$

We introduce

$$\begin{aligned} E_+(p) &= E(p) && \text{if} \quad |p| \leqslant \delta \\ E_+(p) &= \max\{E(p), \gamma\delta^2\} && \text{if} \quad |p| \geqslant \delta \end{aligned} \tag{2.10}$$

and for $\epsilon > 0$ we choose f by

$$f_\epsilon(x) = c_\epsilon(x) + h_\epsilon(x) \tag{2.11}$$

where

$$c_\epsilon(x) = \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu}\,\frac{\cos k.x}{E_+(k) + \epsilon} \tag{2.12}$$

and

$$h_\epsilon(x)=\begin{cases} c_\epsilon(0)-c_\epsilon(x)=\displaystyle\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{1-\cos k.x}{E_+(k)+\epsilon}, & x\in\Lambda \\ 0, & x\notin\Lambda \end{cases} \tag{2.13}$$

Notice that the choice of h_ϵ is such that $f_\epsilon(x)=c_\epsilon(0)$, $\forall x\in\Lambda$, where

$$c_\epsilon(0)=\int\frac{d^\nu k}{(2\pi)^\nu}\,\frac{1}{E_+(k)+\epsilon} \tag{2.14}$$

We estimate $|\tilde{f}_\epsilon(p)|^2$ by

$$|\tilde{f}_\epsilon(p)|^2\leqslant|\tilde{c}_\epsilon(p)|^2+2|\tilde{c}_\epsilon(p)|\,|\tilde{h}_\epsilon(p)|+|\tilde{h}_\epsilon(p)|^2 \tag{2.15}$$

For $\tilde{h}_\epsilon(p)$ the following estimate holds:

$$|\tilde{h}_\epsilon(p)|\leqslant Q(\Lambda)<\infty \tag{2.16}$$

where $Q(\Lambda)$ is a constant depending on Λ, but not on ϵ. In fact, from the definition (2.13)

$$|\tilde{h}_\epsilon(p)|\leqslant\sum_{x\in\Lambda}\left|\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{1-\cos k.x}{E_+(p)+\epsilon}\right|\leqslant\left(\sum_{x\in\Lambda}\frac{|x|^2}{2}\right)\int\frac{d^\nu k}{(2\pi)^\nu}\,\frac{k^2}{E_+(k)}$$

and since $I=\int[d^\nu k/(2\pi)^\nu][k^2/E_+(k)]<\infty$ [because of (2.9)] (2.16) follows with $Q(\Lambda)=I\sum_{x\in\Lambda}(|x|^2/2)$.

Since

$$\tilde{c}_\epsilon(p)=\frac{1}{E_+(p)+\epsilon} \tag{2.17}$$

using (2.8), (2.15), and (2.16) we get

$$\begin{aligned}|\omega(K)|&\leqslant\int\frac{d^\nu p}{(2\pi)^\nu}\,\frac{1}{E_+(p)+\epsilon}+2Q(\Lambda)+Q(\Lambda)^2\int_{B_\nu}\frac{d^\nu p}{(2\pi)^\nu}\,E(p)\\ &=c_\epsilon(0)+D(\Lambda)\end{aligned} \tag{2.18}$$

where

$$D(\Lambda)=2Q(\Lambda)+Q(\Lambda)^2\int_{B_\nu}\frac{d^\nu p}{(2\pi)^\nu}\,E(p)<\infty$$

is independent of ϵ.

Therefore, from (2.7) and (2.18)

$$\left|\frac{d}{ds}\bigg|_{s=0}\omega(\sigma_s A)\right|^2\leqslant\beta\omega\left(\frac{A^*A+AA^*}{2}\right)\frac{c_\epsilon(0)+D(\Lambda)}{c_\epsilon(0)^2} \tag{2.19}$$

This concludes the proof since

$$\lim_{\epsilon\downarrow 0} c_\epsilon(0) = \lim_{\epsilon\downarrow 0} \int \frac{d^\nu p}{E_+(p)+\epsilon} = \infty \qquad \text{iff} \quad \int_{|p|\leqslant\delta} \frac{d^\nu p}{E(p)} = \infty$$

Remarks. Since

$$E(p) = \sum g(x)(1-\cos p.x)$$

with $g(x) \geqslant 0$ we see that for $\nu = 1$ if $x_0 \neq 0$ is such that $g(x_0) > 0$ then $E(p) \geqslant g(x_0)(1-\cos px_0) \geqslant (2g(x_0)/\pi^2)|x_0|^2 p^2$ for $|p| \leqslant \pi/|x_0|$. [Of course, if $g(x) = 0$, $\forall x \in \mathbb{Z}^\nu$, then automatically $(d/ds)\omega(\sigma_s A)|_{s=0} = 0$.] For $\nu = 2$ then *either* there are $x_0 = (x_0^1, x_0^2)$ and $y_0 = (y_0^1, y_0^2)$ with $x_0^1 \neq 0$ and $y_0^2 \neq 0$ (x_0 may be equal to y_0) such that $g(x_0) \neq 0$ and $g(y_0) \neq 0$ and so

$$E(p) \geqslant g(x_0)(1-\cos p.x_0) + g(y_0)(1-\cos p.y_0) \geqslant c|p|^2$$

in some neighborhood of $p = 0$ *or* then the problem can be reduced to the one-dimensional case. In all cases we verify condition (2.9).

2.2. Cluster Properties

In this section we show how the methods of Ref. 1 and of the previous section can be used in the derivation of bounds and cluster properties of certain correlation functions. In general we do not expect clustering for all correlation functions as there are models in two dimensions[13] with short-range interactions and a continuous symmetry exhibiting long-range order.

Without loss of generality we shall consider two regions $\Lambda_0 \ni 0$ and $\Lambda_R \ni R$, with $\Lambda_0 \cap \Lambda_R = \emptyset$, and three observables $A, D \in \mathcal{Q}_{\Lambda_0}$, $B_R \in \mathcal{Q}_{\Lambda_R}$ such that

$$A = \frac{d}{ds}\sigma_s D|_{s=0} \tag{2.20}$$

i.e., in terms of local generators,

$$A = i\left[J_{\Lambda_0}, D\right] \tag{2.20$'$}$$

The key point of our analysis is the identity

$$\omega(AB_R) = \frac{(d/ds)\omega(\sigma_s(f)(DB_R))|_{s=0}}{f(0)} \tag{2.21}$$

provided f is chosen such that

$$f(x) = \begin{cases} f(0) \neq 0, & x \in \Lambda_0 \\ 0, & x \in \Lambda_R \end{cases} \tag{2.22}$$

and arbitrary otherwise. In other words the action of the group is constant in Λ_0 and is the identity in Λ_R.

The following choice of f is convenient:

$$f_R(x) = c_R(x) + h_R(x) \tag{2.23}$$

where

$$c_R(x) = \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{\cos k.x - \cos k.(x-2R)}{E_+(k)} = \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{1-e^{-2ik.R}}{E_+(k)} e^{ik.x} \tag{2.24}$$

and

$$h_R(x) = \begin{cases} c_R(0) - c_R(x), & x \in \Lambda_0 \\ -c_R(x), & x \in \Lambda_R \\ 0, & \text{otherwise} \end{cases} \tag{2.25}$$

With the above choice condition (2.22) is met. The choice of c_R was inspired by Ref. 10.

Bogoliubov's inequality together with (2.21) yields

$$|\omega(AB_R)|^2 \leqslant \beta \|B_R\| \, \|D\| \frac{\omega(K)}{|f_R(0)|^2} \tag{2.26}$$

Assuming again (i), (ii), and (iii) of Theorem 1 we have

$$|\omega(K)| \leqslant \int_{B_\nu} \frac{d^\nu p}{(2\pi)^\nu} |\tilde{f}_R(p)|^2 E(p) \tag{2.27}$$

Lemma. The following estimate holds:

$$|\tilde{h}_R(p)| \leqslant Q(\Lambda_0, \Lambda_R) c_R(0)^{1/2}$$

where $Q(\Lambda_0, \Lambda_R) < \infty$ is a constant depending on the sizes of Λ_0, Λ_R (but not on R).

Proof. From the definition of h_R,

$$\tilde{h}_R(p) = \sum_{x \in \Lambda_0} e^{-ip.x} \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{(1-e^{-2ik.R})(1-e^{ik.x})}{E_+(k)}$$

$$+ \sum_{x \in \Lambda_R} e^{-ip.x} \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{\cos k.x - \cos k.(x-2R)}{E_+(k)}$$

Using Schwarz inequality and the identity $\cos a - \cos b = 2\sin[(a+b)/2]\sin[(b-a)/2]$ we obtain

$$|\tilde{h}_R(p)| \leqslant 2 \sum_{x\in\Lambda_0}\left[\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{1-\cos 2k.R}{E_+(k)}\right]^{1/2}$$

$$\times\left(\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{1-\cos k.x}{E_+(k)}\right)^{1/2}$$

$$+2\sum_{x\in\Lambda_R}\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{|\sin[k.(x-R)]\sin(k.R)|}{E_+(k)}$$

$$\leqslant\sqrt{2}\,c_R(0)^{1/2}\left(\sum_{x\in\Lambda_0}|x|\right)I^{1/2}$$

$$+2\int_{B_\nu}\frac{d^\nu k}{(2\pi)^\nu}\,\frac{|k|\,|\sin k.R|}{E_+(k)}\left(\sum_{x\in\Lambda_R}|x-R|\right)$$

where $I=\int_{B_\nu}[d^\nu k/(2\pi)^\nu][k^2/E_+(k)]<\infty$ as in Theorem 1. Since $\sum_{x\in\Lambda_0}|x|\leqslant(\operatorname{diam}\Lambda_0)|\Lambda_0|$, $\sum_{x\in\Lambda_R}|x-R|\leqslant(\operatorname{diam}\Lambda_R)|\Lambda_R|$, where $\operatorname{diam}\Lambda=\max_{x,y\in\Lambda}|x-y|$ and $|\Lambda|$ is the "volume" of Λ, applying once more Schwarz inequality we finally obtain

$$|\tilde{h}_R(p)|\leqslant\sqrt{2}\left[(\operatorname{diam}\Lambda_0)|\Lambda_0|+(\operatorname{diam}\Lambda_R)|\Lambda_R|\right]I^{1/2}c_R(0)^{1/2}$$

which proves the lemma with

$$Q(\Lambda_0,\Lambda_R)=\sqrt{2}\left[(\operatorname{diam}\Lambda_0)|\Lambda_0|+(\operatorname{diam}\Lambda_R)|\Lambda_R|\right]I^{1/2}. \quad \blacksquare$$

From (2.27) and the lemma we estimate

$$|\omega(K)|\leqslant\int_{B_\nu}\frac{d^\nu p}{(2\pi)^\nu}E(p)\left[|\tilde{c}_R(p)|^2+2|\tilde{c}_R(p)|\,|\tilde{h}_R(p)|+|\tilde{h}_R(p)|^2\right]$$

$$\leqslant 2c_R(0)+2\sqrt{2}\,Q(\Lambda_0,\Lambda_R)c_R(0)^{1/2}\int_{B_\nu}\frac{d^\nu p}{(2\pi)^\nu}(1-\cos 2p.R)^{1/2}$$

$$+Q^2(\Lambda_0,\Lambda_R)c_R(0)\int_{B_\nu}\frac{d^\nu p}{(2\pi)^\nu}E(p)$$

Since both integrals in the above expression are finite, uniformly in R, it follows that

$$|\omega(K)|\leqslant a(\Lambda_0,\Lambda_R)c_R(0)+b(\Lambda_0,\Lambda_R)c_R(0)^{1/2} \tag{2.28}$$

where a and b are constants independent of R.

Theorem 2. Let $j(x, y)$ satisfy properties (i), (ii), and (iii) of Theorem 1. If $A \in \mathfrak{A}_{\Lambda_0}$, $B \in \mathfrak{A}_{\Lambda_R}$, with $\Lambda_0 \cap \Lambda_R = \varnothing$ and $A = (d/ds)(\sigma_s D)|_{s=0}$ for some $D \in \mathfrak{A}_{\Lambda_0}$, then

$$|\omega(AB_R)|^2 \leqslant \|B_R\| \, \|D\| \left| \frac{a(\Lambda_0, \Lambda_R)}{c_R(0)} + \frac{b(\Lambda_0, \Lambda_R)}{c_R(0)^{3/2}} \right|$$

where $a(\Lambda_0, \Lambda_R)$ and $b(\Lambda_0, \Lambda_R)$ are constants depending on Λ_0, Λ_R but not on R.

Proof. Immediate after (2.26) and (2.28).

Remarks. (1) The clustering of $\omega(AB_R)$ is implied by the fact that $c_R(0) \to \infty$ as $|R| \to \infty$ if $\int_{|p|<\delta} d^\nu p E(p)^{-1} = \infty$ (see Lemma A.2 in the Appendix). In this case for large enough $|R|$, $c_R(0) > c_R(0)^{1/2}$, and we can rewrite the bound of Theorem 2 in a simpler form:

$$|\omega(AB_R)|^2 \leqslant \beta \|B_R\| \, \|D\| \frac{a'(\Lambda_0, \Lambda_R)}{c_R(0)}$$

(2) For one-dimensional lattice systems the results of Dobrushin[14,15] imply L^1 clustering if $\sum |g(x)| \, |x| < \infty$ (here g is the coupling). Therefore our results are weaker in this case but are new in the cases where $\sum_{|x|<R} g(x)|x|$ has logarithmic divergencies (see following section).

2.3. Applications

The discussion of this section is to provide a content for Theorems 1 and 2. In fact we shall exhibit classes of models for which conditions (i), (ii), and (iii) are verified. Our discussion is based on Ref. 1.

We assume that to each finite region $\Lambda \subset \mathbb{Z}^\nu$ is associated the $|\Lambda|$-body interaction $H(\Lambda) \in \mathfrak{A}_\Lambda$, such that for each $x \in \mathbb{Z}^\nu$, $\sum_{\Lambda \ni x} \|H(\Lambda)\| < \infty$.

The Hamiltonian is formally defined by

$$H = \sum_{\Lambda \subset \mathbb{Z}^\nu} H(\Lambda)$$

in the sense that for $A \in \mathfrak{A}_\Gamma$, $\Gamma \subset \mathbb{Z}^\nu$,

$$[H, A] = \sum_{\Gamma \cap \Lambda \neq \varnothing} [H(\Lambda), A]$$

is well defined (the series is norm convergent in $\mathfrak{A}$).

Also

$$\frac{d}{ds} \sigma_s(x) H \Big|_{s=0} = \sum_{\Lambda \ni x} \frac{d}{ds} \sigma_s(x) H(\Lambda) \Big|_{s=0}$$

which in terms of local generators is

$$\frac{d}{ds}\sigma_s(x)H\Big|_{s=0} = \sum_{\Lambda \ni x} i\big[J(x), H(\Lambda)\big]$$

and the corresponding expression for the function $j(x, y)$ is

$$j(x,y) = \sum_{\Lambda \ni x,y} \omega\big(\big[J(x),\big[H(\Lambda),J(y)\big]\big]\big)$$

Condition (i) is a consequence of $[\sigma_s(x),\sigma_s(y)] = 0$. The first part of condition (ii) follows from the bound:

$$|j(x,y)| \leqslant 4\|J(x)\|\,\|J(y)\| \sum_{\Lambda \ni x,y} \|H(\Lambda)\|$$

so that

$$\|j(x,.)\|_1 = \sum_{y \ni \mathbb{Z}^\nu} |j(x,y)| \leqslant \text{const}\|J(x)\| \sum_{\Lambda \ni x} |\Lambda|\,\|H(\Lambda)\|$$

where we assumed that $\sup_{x\in\mathbb{Z}^\nu}\|J(x)\| < \infty$ and $\sum_{\Lambda\ni x}|\Lambda|\,\|H(\Lambda)\| < \infty$ for each $x \in \mathbb{Z}^\nu$.

In particular if the interaction is at most N body, that is, if $H(\Lambda) = 0$ if $|\Lambda| > N$, then

$$\|j(x,.)\|_1 \leqslant \text{const} \sum_{\Lambda \ni x} \|H(\Lambda)\| < \infty$$

The second part of (ii) is a statement about the invariance of $H(\Lambda)$ under the symmetry σ_s:

$$\sigma_s H(\Lambda) = H(\Lambda) \qquad \text{for all} \quad \Lambda \subset \mathbb{Z}^\nu, \quad \text{which implies}$$

$$\sum_y j(x,y) = \sum_{\Lambda \ni x} \omega\left(\left[J(x), \frac{d}{ds}\sigma_s H(\Lambda)\Big|_{s=0}\right]\right) = 0$$

Finally condition (iii) will follow from the uniform bound:

$$\begin{aligned} |j(x,y)| &\leqslant 4\|J(x)\|\,\|J(x-z)\| \sum_{\Lambda \ni x, x-z} \|H(\Lambda)\| \\ &\leqslant g(z) \qquad \text{for any} \quad x,y \in \mathbb{Z}^\nu, \quad \text{where} \quad z = x-y. \end{aligned}$$

In the case $\|J(x)\| = \|J(0)\|$, for all $x \in \mathbb{Z}^\nu$, and $H(\Lambda)$ is translation invariant the function $g(x-y)$ may be defined by

$$g(x-y) = 4\|J(0)\|^2 \sum_{\Lambda \ni 0, x-y} \|H(\Lambda)\|$$

Notice that no assumption is made concerning translation invariance either of the state $\omega(\cdot)$ or of the Hamiltonian in general.

As a more concrete application we shall now consider the simpler case of the Heisenberg model defined in the lattice (with two-body interaction).

To each site $x \in \mathbb{Z}^\nu$ there correspond spin operators $S_1(x)$, $S_2(x)$, $S_3(x)$ with the usual commutation relations, and with $\sum_{i=1}^3 S_i^2(x) = S(S+1)$.

The Heisenberg Hamiltonian is given (formally) by

$$H = \sum_{x,y \in \mathbb{Z}^\nu} \{ I_1(x,y)[S_1(x)S_1(y) + S_2(x)S_2(y)] + I_2(x,y)S_3(x)S_3(y) \}$$

with $I_1(x,y) = I_1(y,x)$.

The symmetry group consists of rotations around axis 3, which local generator is $S_3(x)$.

With the notation of Section 2.2 we choose

$$\Lambda_0 = \{0\}, \qquad \Lambda_R = \{R\}, \qquad D = S_2(0), \qquad B_R = S_1(R)$$

and Theorem 2 reads

$$|\omega(S_1(0)S_1(R))|^2 \leqslant \beta S^2 \frac{\text{const}}{c_R(0)} \qquad \text{for} \quad |R| \text{ large}$$

The denominator $c_R(0)$ is given, as before, by

$$c_R(0) = \int_{B_\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{1 - \cos 2k.R}{E(k)}$$

with

$$E(k) = 4S^2 \sum_{x \in \mathbb{Z}^\nu} (1 - \cos k.x) g(x)$$

where $g(x)$ is such that $|I_1(x,y)| \leqslant g(x-y)$.

We assume here that $g(x)$ is such that $E(k)$ has no other zeros than $k = 0$, so we can take $E_+(k) = E(k)$, without worrying about other singularities of $c_R(0)$.

Taking $|R| \to \infty$ implies $c_R(0) \to \infty$, provided $\int_{B_\nu} [d^\nu k / E(k)] = \infty$ (Lemma A2 of the Appendix), the rate of divergence depending on the singularities of $E(k)^{-1}$ at $k = 0$.

If we have

$$\sum_{x \in \mathbb{Z}^\nu} |x|^\nu g(x) = \alpha < \infty$$

then

$$E(k) \leqslant 2S^2 |k|^\nu \sum_{x \in \mathbb{Z}^\nu} |x|^\nu g(x) = 2\alpha S^2 |k|^\nu$$

and

$$c_R(0) \geqslant \frac{2\alpha S^2}{(2\pi)^\nu} \int_{B_\nu} \frac{1 - \cos 2k.R}{|k|^\nu} d^\nu k \sim \ln|R|, \qquad \text{for } |R| \text{ large.}$$

If the ν-moment of $g(x)$ has only logarithmic divergencies, that is, for some $m \geqslant 1$

$$\sup_Q \frac{1}{\ln Q \ln_2 Q \ldots \ln_m Q} \sum_{|x|<Q} |x|^\nu g(x) < \infty$$

where $\ln_k Q = \ln \ln \ldots \ln Q$ (k times), the behavior of $E(k)$ for $|k|$ sufficiently small will be

$$E(k) \leqslant C|k|^\nu \ln|k|^{-1} \ln_2|k|^{-1} \ldots \ln_m|k|^{-1}$$

(for a proof see the Appendix) and $c_R(0) \sim \ln_{m+1}|R|$, $|R|$ large.

3. CONTINUUM SYSTEMS

In this section we briefly discuss as our results and techniques can be extended to cover classical and quantum systems defined on the continuum $\mathbb{R}^\nu$. We shall not discuss in this paper the features of the interaction and of the states necessary for the assumptions involved to be valid.

Continuum systems are also described by a C^* algebra $\mathcal{Q} = \overline{\bigcup_{\Lambda \subset \mathbb{R}^\nu, \Lambda \text{ bounded}} \mathcal{Q}_\Lambda}$ and we will also assume the existence, in the reconstructed GNS Hilbert space, of local generators (in general unbounded operators) $J(x)$, $x \in \mathbb{R}^\nu$, of the symmetry, i.e.,

$$\sigma_s(f)A = e^{isJ(f)} A e^{-isJ(f)} \qquad \text{for} \quad A \in \mathcal{Q}_\Lambda$$

where $J(f) = \int d^\nu x f(x) J(x)$ with $f(x) = 1$ for $x \in \Lambda$ [$J(x)$ need not be strictly localized; see Ref. 16].

As in Section 2.1 we define

$$\omega(K) = \int \int d^\nu x \, d^\nu y \, f(x) f(y) j(x, y)$$

with

$$j(x, y) = -i\omega\left(\left[J(x), \dot{J}(y)\right]\right) \quad \text{and} \quad \dot{J}(y) = \frac{d}{dt} e^{itH} J(y) e^{-itH}\Big|_{t=0}$$

Theorem 3. Any state ω in a continuum system satisfying Bogoliubov's inequality and

(i) $j(x, y)$ is measurable and $j(x, y) = j(y, x)$ a.e.,
(ii) $j(x, .) \in L^1(\mathbb{R}^\nu)$ and $\int_{\mathbb{R}^\nu} j(x, y) \, d^\nu y = 0$ a.e.,
(iii) there exists a function $g \in L^1(\mathbb{R}^\nu)$ such that

$$|j(x, y)| \leqslant g(x - y) \text{ a.e.} \qquad \text{and}$$

$$\int_{|p|<\delta} \frac{d^\nu p}{E(p)} = \infty \qquad \text{for all} \quad \delta > 0$$

where $E(p) = \int(1 - \cos p.x)g(x)\,d^\nu x$, is invariant under the symmetry group.

The proof is entirely analogous to that of Theorem 1, with a slightly different choice of the function $c_\epsilon(x)$:

$$c_\epsilon(x) = \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu}\,\frac{\cos k.x}{E(k)+\epsilon}\,\phi(k) \tag{3.1}$$

with $\phi(k) \in C_0^\infty(\mathbb{R}^\nu)$, $\phi(k) = \phi(-k)$, $\phi(k) = 0$ for $|k| > \delta$, for some $\delta > 0$ and $\phi(0) = 1$.

Here we need not define $E_+(k)$ as in (2.10), due to the introduction of the large-k cutoff $\phi(k)$ in (3.1). Also notice that for any $\epsilon > 0$, $c_\epsilon(x)$ decreases exponentially fast as $|x| \to \infty$, which will ensure a "bona-fide" definition of $J(f)$ in most cases.

The continuum analog of Theorem 2 is the following:

Theorem 4. Let $A \in \mathscr{Q}_{\Lambda_0}$, $B \in \mathscr{Q}_{\Lambda_R}$, with $\Lambda_0 \cap \Lambda_R = \varnothing$ and $A = (d/ds)(\sigma_s D)|_{s=0}$ for some $D \in \mathscr{Q}_{\Lambda_0}$. Then for any state in a continuum system satisfying the hypothesis of Theorem 3 we have

$$|\omega(AB_R)|^2 \leqslant \beta\|D\|\,\|B_R\|\left[\frac{Q_1(\Lambda_0,\Lambda_R)}{c_R(0)} + \frac{Q_2(\Lambda_0,\Lambda_R)}{c_R(0)^{3/2}}\right]$$

where $Q_1(\Lambda_0,\Lambda_R)$ and $Q_2(\Lambda_0,\Lambda_R)$ are constants depending on Λ_0, Λ_R but not on R, and where

$$c_R(0) = \int \frac{d^\nu k}{(2\pi)^\nu}\,\frac{1-\cos 2k.R}{E(k)}\,\phi(k)$$

with $\phi(k)$ defined above.

Again the proof is similar to that of Theorem 2, where now

$$c_R(x) = \int \frac{d^\nu k}{(2\pi)^\nu}\,\frac{\cos(k.x) - \cos[k.(x-2R)]}{E(k)}\,\phi(k)$$

Here also, by Lemma A2, $\lim_{|R|\to\infty} c_R(0) = \infty$ if $\int_{|k|<\delta}[d^\nu k/E(k)] = \infty$, which gives clustering for $\omega(AB_R)$.

As a final remark we notice that Theorems 3 and 4 are also valid for classical systems in the continuum case, replacing commutators by Poisson brackets.

APPENDIX

In this section we extend some results contained in Ref. 8 and, for the reader's convenience, we give a simple proof of the divergent behavior of $c_R(0)$ as $|R| \to \infty$.

Let $\Lambda_N \subset \mathbb{Z}^\nu$, $\nu = 1$ or 2, be the "square" centered at the origin with sides $2N$, N integer, that is, $\Lambda_N = \{-N, -N+1, \ldots, N\}^\nu$.

Lemma A1. Let $E(p) = \sum_{x\in\mathbb{Z}^\nu}(1-\cos p.x)g(x)$ with $g(x) \geqslant 0$ and let $K(N) = \sum_{x\in\Lambda_N}|x|^\nu g(x)$. If $\sup_N[K(N)/\ln N \ln_2 N \ldots \ln_k N] < \infty$ for some $k \geqslant 1$ then, for $|p|$ sufficiently small,

$$E(p) \leqslant C|p|^\nu \ln|p|^{-1}\ln_2|p|^{-2} \ldots \ln_k|p|^{-1}$$

Proof. The proof is along the lines of that of Theorem 5.5 in Ref. 8. Since $1-\cos p.x \leqslant (1/2)|p|^2|x|^2$ and $1-\cos p.x \leqslant |p|\,|x|$,

$$\begin{aligned} E(p) &= \sum_{x\in\Lambda_N}(1-\cos p.x)g(x) + \sum_{x\in\Lambda_N^c}(1-\cos p.x)g(x) \\ &\leqslant |p|^\nu \sum_{x\in\Lambda_N}|x|^\nu g(x) + 2\sum_{x\in\Lambda_N^c} g(x) \end{aligned}$$

For $M > N$

$$\sum_{x\in(\Lambda_M\setminus\Lambda_N)} g(x) = \sum_{n=N+1}^{M}\ \sum_{x\in\partial\Lambda_n}\frac{|x|^\nu}{|x|^\nu}g(x) \leqslant \sum_{n=N+1}^{M}\frac{1}{n^\nu}\sum_{x\in\partial\Lambda_n}|x|^\nu g(x) \tag{A1}$$

as $|x|^\nu \geqslant n^\nu$ for $x \in \partial\Lambda_n$. Now

$$\sum_{x\in\partial\Lambda_n}|x|^\nu g(x) = K(n) - K(n-1)$$

and so (A1) is bounded by

$$\begin{aligned} \sum_{x\in(\Lambda_M\setminus N_N)} g(x) &\leqslant \sum_{n=N+1}^{M}\frac{1}{n^\nu}\left[K(n)-K(n-1)\right] \\ &= \sum_{n=N+1}^{M-1}\left[\frac{1}{n^\nu}-\frac{1}{(n+1)^\nu}\right]K(n) \\ &\quad + \frac{1}{M^\nu}K(M) - \frac{1}{(N+1)^\nu}K(N) \\ &\leqslant \frac{1}{M^\nu}K(M) + 3\sum_{n=N+1}^{M-1}\frac{K(n)}{n^{\nu+1}} \end{aligned} \tag{A2}$$

But

$$K(n) \leqslant C \ln n \ln_2 n \ldots \ln_k n \tag{A3}$$

and hence

$$\lim_{M\to\infty} \frac{K(M)}{M^{\nu}} = 0 \tag{A4}$$

Taking the limit $M \to \infty$ in (A2) and using (A3) and (A4) gives

$$\sum_{x\in\Lambda_N^c} g(x) \leqslant 3C \sum_{n=N+1}^{\infty} \frac{\ln n \ln_2 n \ldots \ln_k n}{n^{\nu+1}} \leqslant 3C \int_N^\infty \frac{\ln x \ln_2 x \ldots \ln_k x}{x^{\nu+1}}\, dx$$

Integration by parts yields

$$\begin{aligned}\int_N^\infty &\frac{\ln x \ln_2 x \ldots \ln_k x}{x^{\nu+1}}\, dx \\ &= \frac{1}{\nu}\frac{1}{N^\nu} \ln N \ln_2 N \ldots \ln_k N \\ &\quad + \frac{1}{\nu}\int_N^\infty \frac{1}{x^{\nu+1}} (\ln_2 x \ldots \ln_k x + \ln_3 x \ldots \ln_k x + \cdots + 1)\, dx \end{aligned} \tag{A5}$$

If N is sufficiently large

$$\ln N \geqslant \ln_2 N \geqslant \cdots > \ln_k N > 2$$

which leads to

$$\int_N^\infty \frac{\ln x \ln_2 x \ldots \ln_k x}{x^{\nu+1}}\, dx \leqslant \frac{2}{2\nu - 1}\frac{1}{N^\nu} \ln N \ln_2 N \ldots \ln_k N$$

By choosing $N = [|p|^{-1}]$ we conclude the proof. ■

Remark. For the one-dimensional case ($\nu = 1$) better bounds can be obtained[1] for particular $g(x)$. For instance, if $g(x) = C/(1 + x^2)$ then

$$\begin{aligned}\sum_{x=-\infty}^{\infty} &(1 - \cos p.x) g(x) \\ &= 4C \sum_{x=1}^{\infty} \frac{\sin^2(px/2)}{1+x^2} \leqslant 4C\int_0^\infty \frac{\sin^2(px/2)}{1+x^2}\, dx \\ &= 4C|p|\int_0^\infty \frac{\sin^2(y/2)}{p^2+y^2}\, dy \leqslant 4C|p|\int_0^\infty \frac{\sin^2(y/2)}{y^2}\, dy \leqslant \text{const}|p| \end{aligned}$$

which improves the bound from Lemma A1.

Lemma A2. Let $G(x)$ be a continuous function on $B_\nu - \{0\}$ such that $G(x) \geqslant 0$ and $\int_{B_\nu} G(x)\, d^\nu x = \infty$. Then

$$\lim_{R\to\infty} \int_{B_\nu} (1 - \cos 2R.x) G(x)\, d^\nu x = \infty$$

Proof. Let $|R|_\infty = \max\{R^1, \ldots, R^\nu\}$. For $R \neq 0$ let us divide $\mathbb{R}^\nu$ into cubic regions with sides of length $\pi/|R|_\infty$, such that $x = 0$ is the center of one such region, and let I_j, $j = 1, \ldots, N(R)$, be those cubes contained in B_ν, not including the cube centered at $x = 0$. Then

$$\begin{aligned}\int_{B_\nu}(1-\cos 2R.x)G(x)d^\nu(x) &\geqslant \sum_{j=1}^{N(R)}\int_{I_j}(1-\cos 2R.x)G(x)\,d^\nu x\\ &\geqslant \sum_{j=1}^{N(R)}\Big[\min_{x\in I_j}G(x)\Big]\int_{I_j}(1-\cos 2R.x)\,d^\nu x\\ &= \sum_{j=1}^{N(R)}|I_j|\Big[\min_{x\in I_j}G(x)\Big]\end{aligned}$$

since $\int_{I_j}\cos 2R.x\,d^\nu x = 0$.

But as $R \to \infty$ we have that $|R|_\infty \to \infty$ and hence

$$\sum_{j=1}^{N(R)}|I_j|\Big[\min_{x\in I_j}G(x)\Big] \to \int_{B_\nu}G(x)\,d^\nu x = \infty \quad \blacksquare$$

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